MATH 2010B Advanced Calculus I Lecture Notes (Week 1)

Outline of the Course

Martin Li

(1) Single variable calculus.

$f : \mathbb{R} \to \mathbb{R}$

$f(x)$

e.g. $f(x) = \cos x$

$= x^2$.

differentiation. $\frac{df}{dx}$, $f'(x)$, chain rule, Taylor's Thm.

Implicit diff., tangent lines, slopes.

integration. $\int f\,dx$, $\int_a^b f(x)\,dx$, areas under curve.

u-sub., integration by part. etc.

Fundamental Theorem(s) of Calculus:

(i) $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$

(ii) $\displaystyle\frac{d}{dx}\left(\int_a^x f(y)\,dy\right) = f(x)$.

applications

- min/max $f(x)$.

---

(2) Multivariable Calculus.

$F : \mathbb{R}^n \to \mathbb{R}^m$

functions of several variables.

$F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$.

Q1: How to differentiate (MATH 2010) & integrate (MATH 2020). F ?

Q2: Is there a fundamental theorem of calculus ? (MATH 2020).

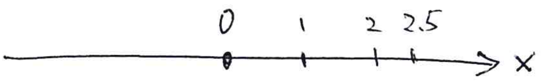
Q3: min/max $F(x_1, \ldots, x_n)$.

subject to ~~constraints~~. constraints.

Euclidean Space ($\mathbb{R}^n$)

$$\mathbb{R}^n := \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies}} = \{ \overset{n\text{-tuple.}}{(x_1, \dots, x_n)} \mid x_i \in \mathbb{R} \quad i = 1, \dots, n \}.$$

<u>$n=1$</u>: $\mathbb{R}^n = \mathbb{R}$ real line.



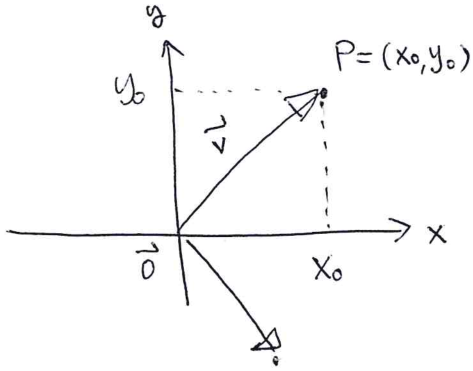
<u>Properties of $\mathbb{R}$</u>

(a) $(+, -, \times, \div)$ arithmetic.

(b) either $x < 0$, $x = 0$ or $x > 0$.

(ordering).

<u>$n=2$</u>: $\mathbb{R}^2$ plane.



$(x_0, y_0) \in \mathbb{R}^2$

$\Leftrightarrow$ $P = (x_0, y_0)$ a <u>point</u> on the plane with coordinates x_0 & y_0.

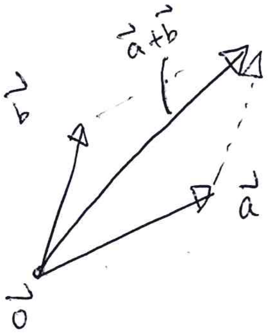
$\Leftrightarrow$ $\vec{v}$ as a <u>vector</u> from $\vec{0}$ to P.

Given $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2) \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

<u>Q</u>: what can we do to vectors?

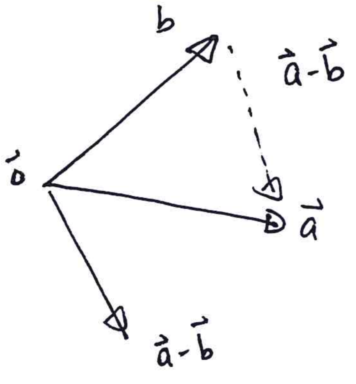
<u>Equality</u>: $\vec{a} = \vec{b}$ means $a_1 = b_1$, and $a_2 = b_2$.

<u>Addition</u>: $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2)$

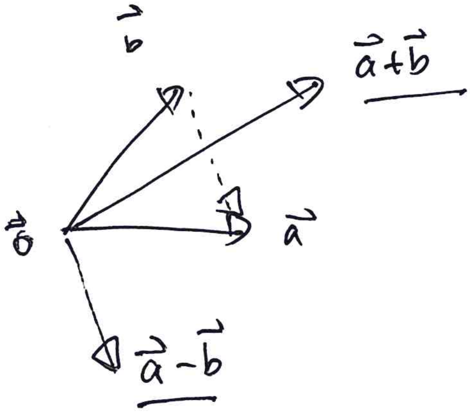


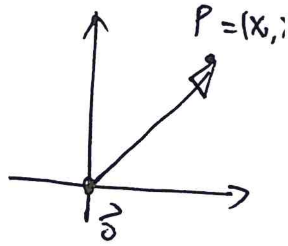
parallelogram law.

<u>Subtraction</u>: $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2$



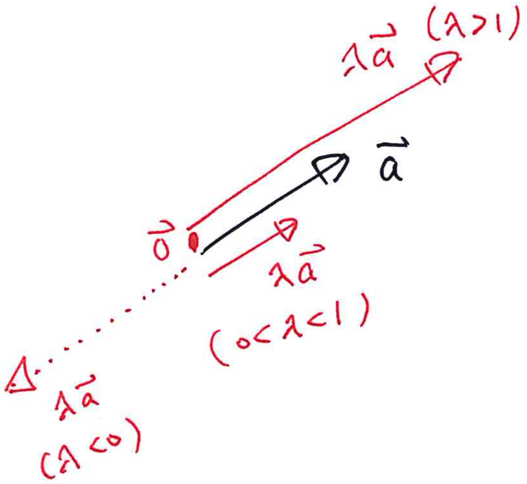
Last time. $\mathbb{R}^n$: $\vec{x} = (x_1, \dots, x_n) \longleftrightarrow$ point / vector.





Scalar Multiplication (Rescaling).

Given $\vec{a} = (a_1, a_2)$, $\lambda \in \mathbb{R}$



$$\boxed{\lambda\vec{a} := (\lambda a_1, \lambda a_2)}$$

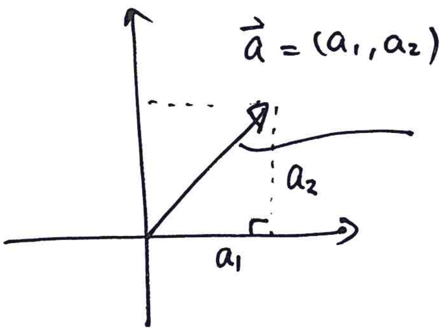
Summary:

- $|\lambda| > 1 \iff$ streching
- $|\lambda| < 1 \iff$ shrinking.
- $\lambda > 0 \implies$ same direction as $\vec{a}$
  $\lambda < 0 \implies$ opposite " " "

(Ex: $\vec{a} - \vec{b} = \vec{a} + (-1) \cdot \vec{b}$)

Length / Norm : $\|\vec{a}\| := \sqrt{a_1^2 + a_2^2}$

$\vec{a} = (a_1, a_2)$



length $= \sqrt{a_1^2 + a_2^2}$

(Pythagoras Thm.).

Higher dimension $\mathbb{R}^n$, $n \geq 3$.

$$\mathbb{R}^n := \{ (x_1, \dots, x_n) \mid x_i \in \mathbb{R} \}.$$

$$(\mathbb{R}^n, +, \cdot) \qquad \begin{cases} (x_1, \dots, x_n) + (y_1, \dots, y_n) := (x_1+y_1, \dots, x_n+y_n) \\ \lambda (x_1, \dots, x_n) := (\lambda x_1, \dots, \lambda x_n) \end{cases}$$

Fact: $(\mathbb{R}^n, +, \cdot)$ is a vector space (over $\mathbb{R}$).

i.e.

$$(+) \begin{cases} \vec{a} + \vec{b} = \vec{b} + \vec{a} \\ (\vec{a}+\vec{b})+\vec{c} = \vec{a} + (\vec{b}+\vec{c}) \\ \vec{a} + \vec{0} = \vec{a} \qquad \text{where } \vec{0} = (0, \dots, 0) \text{ zero vector / origin.} \\ \vec{a} + (-\vec{a}) = \vec{0} \qquad \text{where } -\vec{a} = (-1) \cdot \vec{a}. \end{cases}$$

$$(+, \cdot) \begin{cases} \lambda (\vec{a} + \vec{b}) = \lambda \vec{a} + \lambda \vec{b} \\ 1 \cdot \vec{a} = \vec{a} \\ (\lambda_1 + \lambda_2) \cdot \vec{a} = \lambda_1 \vec{a} + \lambda_2 \vec{a} \\ (\lambda_1 \lambda_2) \cdot \vec{a} = \lambda_1 \cdot (\lambda_2 \cdot \vec{a}) \end{cases}$$

Basis and Orientation

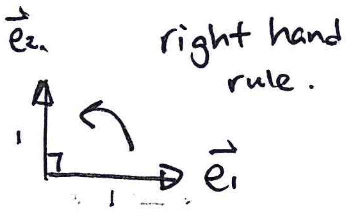
In $\mathbb{R}^n$, let $\vec{e}_i := (0, \dots, 0, 1, 0, \dots, 0)$ $\qquad i = 1, 2, \dots, n$

(1 at the i^{th} coordinate)

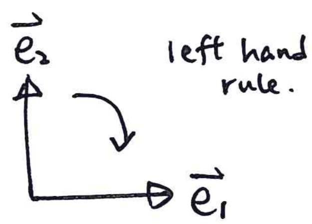
$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ standard ordered basis $\Rightarrow$ standard orientation (right hand rule)

$\underline{n=2}$: $\vec{e}_1 = (1, 0)$

$\vec{e}_2 = (0, 1)$

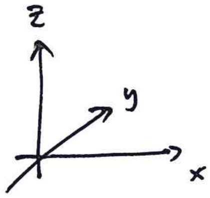


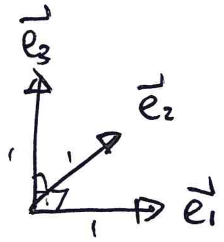
$\{\vec{e}_1, \vec{e}_2\}$



$\{\vec{e}_2, \vec{e}_1\}$ opposite orientation

$\underline{n=3}$:





$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

standard.

Q: Which of these ordered basis gives standard orientation?

(a) $\{\vec{e}_1, \vec{e}_3, \vec{e}_2\}$.

Ex: (b) $\{\vec{e}_2, \vec{e}_3, \vec{e}_1\}$?.

(c) $\{\vec{e}_3, \vec{e}_1, \vec{e}_2\}$.

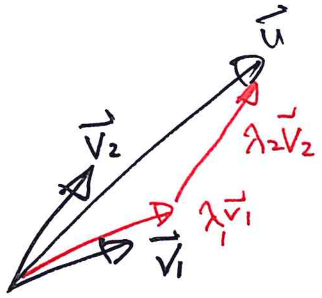
Determinant test.

e.g. (a)

$$\underbrace{\det\begin{pmatrix} - \vec{e}_1 - \\ - \vec{e}_3 - \\ - \vec{e}_2 - \end{pmatrix}}_{3\times 3} = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = -1 < 0 \Rightarrow \text{opposite orientation}$$

(if > 0, $\Rightarrow$ standard orientation)

Linear Algebra: If $\{\vec{v}_1, \vec{v}_2\}$ are linearly independent vectors in $\mathbb{R}^2$, then they form a "basis", i.e. any $\vec{u} \in \mathbb{R}^2$ can be uniquely written as a linear combination of $\vec{v}_1$ & $\vec{v}_2$:



$$\vec{u} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 \quad \text{for some unique } \lambda_1, \lambda_2 \in \mathbb{R}$$

E.g.: $\{\vec{e}_1, \vec{e}_2\}$.

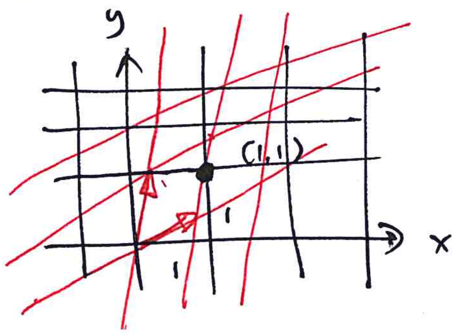
$$\vec{a} = (a_1, a_2)$$
$$= a_1 \cdot \underbrace{(1,0)} + a_2 \cdot \underbrace{(0,1)}$$
$$= a_1 \vec{e}_1 + a_2 \cdot \vec{e}_2$$

Identification:

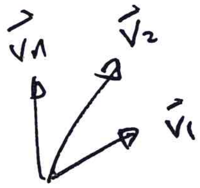
vector $\vec{a}$ $\longleftrightarrow$ $\vec{a} = (a_1, a_2)$

$\uparrow$ depends on the basis.

(change of coordinate).



Orientation of $\{\vec{v}_1, \dots, \vec{v}_n\}$? ordered basis for $\mathbb{R}^n$



det. test works.

$$\det \begin{pmatrix} - \vec{v}_1 - \\ - \vec{v}_2 - \\ \vdots \\ - \vec{v}_n - \end{pmatrix} \begin{cases} > 0 & (\text{standard orientation}) \\ < 0 & (\text{opposite orientation}) \end{cases}$$

$n \times n$

Q: "$= 0$" ? no. because linear indep.

Ex: $\vec{v}_1 = (1,2,3)$
$\vec{v}_2 = (2,3,4)$
$\vec{v}_3 = (4,5,6)$

then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ orientation ?

Recall: $\mathbb{R}^n$, $\vec{a} \pm \vec{b}$, $\lambda \vec{a}$

Q1: Can we multiply $\vec{a} \text{ "}\times\text{" } \vec{b}$ ← vector ? "yes/no". nice properties ?

Q2: Can we divide $\vec{a} \div \vec{b}$?. No! (major difference between number & vector)

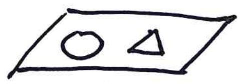
Inner Product $\langle \cdot, \cdot \rangle$

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \leftarrow \text{number.}$$

standard: $\langle \vec{a}, \vec{b} \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \in \mathbb{R}$.

Philosophy: $\langle \cdot, \cdot \rangle$ defines a "geometry" on $\mathbb{R}^n$.

measure distance.

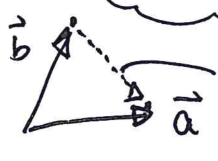


plane

earth.

norm: $\|\vec{a}\| := \langle \vec{a}, \vec{a} \rangle^{1/2}$ (ooo $|a| = \sqrt{a^2}$).

distance between $\vec{a}$ & $\vec{b}$ $= \|\vec{a} - \vec{b}\|$.



length $= \|\vec{a} - \vec{b}\|$.

E.g. (other geometries) in $\mathbb{R}^2$.

(1) $\langle \vec{a}, \vec{b} \rangle := 2a_1b_1 + a_2b_2$

(2) $\langle \vec{a}, \vec{b} \rangle := a_1b_1 - a_2b_2$ (Lorentz geometry) basic foundation of relativity.

Properties of $\langle \cdot, \cdot \rangle$

(1) $\langle \vec{x}, \vec{y}+\vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

$\langle \vec{x}+\vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

(2) $\langle \lambda\vec{x}, \vec{z} \rangle = \lambda \langle \vec{x}, \vec{z} \rangle = \langle \vec{x}, \lambda\vec{z} \rangle$

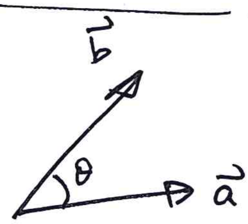
(bilinear) [for (1) and (2)]

(3) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$ (symmetric)

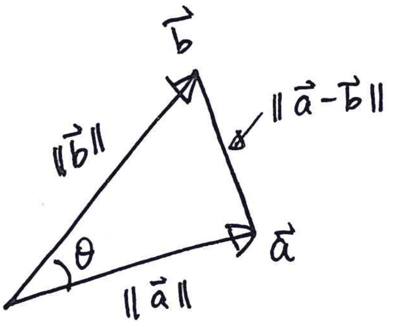
(4) $\underbrace{\langle \vec{x}, \vec{x} \rangle}_{\|\vec{x}\|^2} \geq 0$ and "=" holds $\Leftrightarrow \vec{x} = \vec{0}$. (positive definite).

Angles and Projections

Fact: $\boxed{\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \cos\theta}$ (*)



"Proof"



Cosine Law: (vector)

$$\|\vec{a}-\vec{b}\|^2 = \overbrace{\|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\| \|\vec{b}\| \cos\theta}$$

By def^n,

$$\|\vec{a}-\vec{b}\|^2 = \langle \vec{a}-\vec{b}, \vec{a}-\vec{b} \rangle \quad (\text{def. of } \|\cdot\|)$$

$$= \langle \vec{a}, \vec{a} \rangle - \langle \vec{a}, \vec{b} \rangle - \langle \vec{b}, \vec{a} \rangle + \langle \vec{b}, \vec{b} \rangle$$

$$= \underbrace{\|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\langle \vec{a}, \vec{b} \rangle}$$

Cosine Law $\Rightarrow$ (*).

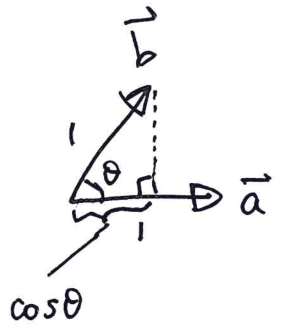
※

Rearrange $(*)$ $\Rightarrow$ $$\cos\theta = \frac{\langle \vec{a}, \vec{b} \rangle}{\underbrace{\|\vec{a}\|\,\|\vec{b}\|}_{\text{defined by } \langle \cdot, \cdot \rangle .}} \qquad (**)$$

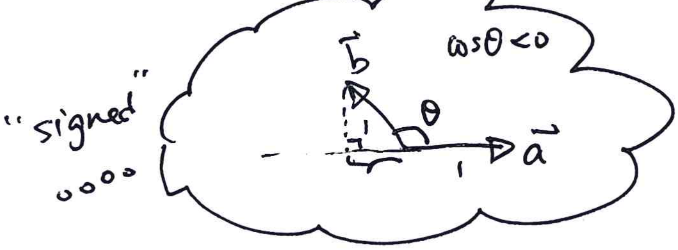
<u>Idea</u>: We can define "angle" between two vectors $\vec{a}, \vec{b}$ $(\neq \vec{0})$ by $(**)$

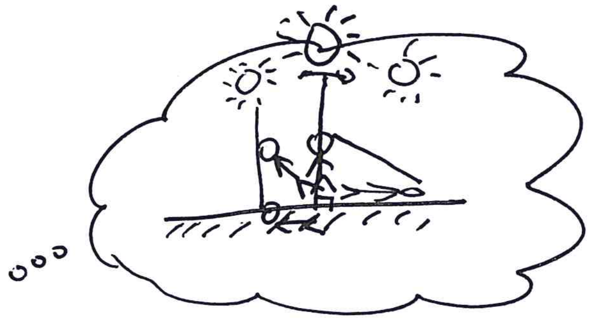
<u>Geometric meaning of $\langle \cdot, \cdot \rangle$</u> (= "projections")

Look at $(**)$., when $\|\vec{a}\| = \|\vec{b}\| = 1$.



$(**)$ $\Rightarrow$ $\cos\theta = \langle \vec{a}, \vec{b} \rangle$ = signed length of orthogonal "$\perp$" projection of $\vec{b}$ onto $\vec{a}$.





<u>Q</u>: What about $\|\vec{a}\|, \|\vec{b}\| \neq 1$?

<u>Def</u>: $\vec{a} \perp \vec{b}$ orthogonal / perpendicular $\iff$ $\boxed{\langle \vec{a}, \vec{b} \rangle = 0}$

(ie $\theta = \frac{\pi}{2}$ or 90°

$\cos\theta = 0$)

Two useful inequalities

(I) Cauchy-Schwartz : $\boxed{|\langle \vec{a}, \vec{b} \rangle| \leq \|\vec{a}\| \|\vec{b}\|}$

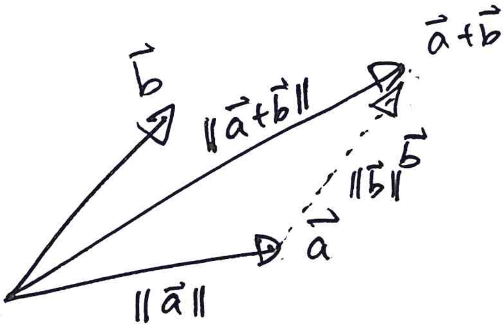
"=" holds $\Leftrightarrow$ $\vec{a} \parallel \vec{b}$ ie. $\vec{b} = \lambda \vec{a}$ for some $\lambda \in \mathbb{R}$.
or $\vec{a} = \lambda \vec{b}$

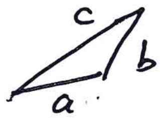
(II) Triangle inequality :

$$\boxed{\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|} \quad \text{—} \ (\#)$$

"=" holds $\Leftrightarrow$ $\vec{a} \parallel \vec{b}$ ie $\vec{b} = \lambda \vec{a}$, $\lambda \geq 0$
same direction

Geometry : (II)





$c \leq a+b$.

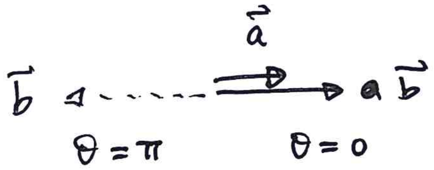
Fact (Ex:) (I) $\Rightarrow$ (II). (idea: square (#) , expand).

Proofs of (I) :

1st "Proof"
(geometric).

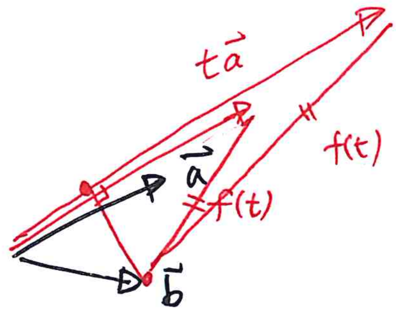
$$(**) \Rightarrow |\cos\theta| = \frac{|\langle \vec{a}, \vec{b} \rangle|}{\|\vec{a}\| \|\vec{b}\|} \leq 1$$

$$\Rightarrow |\langle \vec{a}, \vec{b} \rangle| \leq \|\vec{a}\| \|\vec{b}\|.$$



! This uses $|\cos\theta| \leq 1$!.

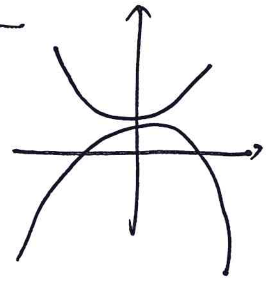
__2nd "Proof"__: Consider the function



$$f(t) := \| t\cdot\vec{a} - \vec{b} \|^2 \geq 0$$

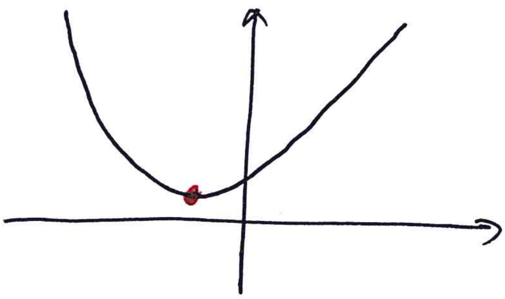
expand:
$$\begin{aligned} f(t) &= \| t\vec{a} - \vec{b} \|^2 \\ &= \langle t\vec{a} - \vec{b}, t\vec{a} - \vec{b} \rangle \\ &= t^2 \langle \vec{a},\vec{a} \rangle - t\underbrace{\langle \vec{a},\vec{b} \rangle - t\langle \vec{b},\vec{a} \rangle}_{=} + \langle \vec{b},\vec{b} \rangle \\ &= \langle \vec{a},\vec{a} \rangle t^2 - 2\langle \vec{a},\vec{b} \rangle t + \langle \vec{b},\vec{b} \rangle. \\ &= \underbrace{\|\vec{a}\|^2 t^2 - 2\langle \vec{a},\vec{b} \rangle t + \|\vec{b}\|^2}_{\text{quadratic polynomial in } t} \end{aligned}$$

$$at^2 + bt + c$$
$$\Delta = b^2 - 4ac$$



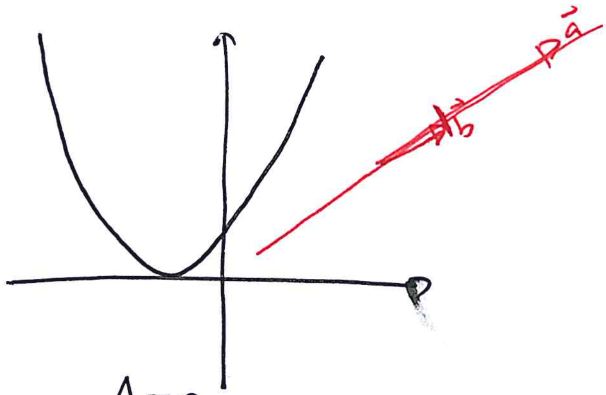
__Note:__ $f(t) \geq 0$ for all t.

graph:



$\Delta < 0$

or



$\Delta = 0$

either case $\Rightarrow \quad \Delta \leq 0$

$\Rightarrow \quad 4\langle \vec{a},\vec{b} \rangle^2 - 4\|\vec{a}\|^2 \|\vec{b}\|^2 \leq 0$

$\Rightarrow \quad \langle \vec{a},\vec{b} \rangle^2 \leq \|\vec{a}\|^2 \|\vec{b}\|^2$

$\downarrow$ sq. root

(I)

__Ex:__ understand this proof geometrically.